The irrationality of $\sqrt{2}$

Mathematics Explained and Clarified

Lemma 1. *For any integer x, if x^2 is even, then x is even as well.*

Proof. Suppose that x is odd. Then $x = 2m + 1$ for some integer m. Then $x^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$. Thus, x^2 is odd, a contradiction. □

Theorem 1. *The number $\sqrt{2}$ is irrational.*

Proof. Suppose that the statement is not true and $\sqrt{2}$ is rational. By definition of a rational number, it means that it can be expressed as p/q, where p and q are integer numbers.

We can assume that p and q are coprime, because if they are not coprime, we can divide both of them by their greatest common divisor and thus obtaining another pair of numbers p and q which are coprime and have the same value of p/q. In particular, what we need is that p and q are not both even at the same time. Either only one of them is even, or both of them are odd.

So, we have

$$\sqrt{2} = \frac{p}{q}.$$

Squaring both sides, we get

$$2 = \frac{p^2}{q^2}.$$

Multiplying by q^2, we get

$$2q^2 = p^2.$$

This means that p^2 is even. By Lemma 1, we conclude that p is even. Then $p = 2m$ for some integer m. Then we have $2q^2 = (2m)^2 = 4m^2$, hence $q^2 = 2m^2$. This means that q^2 is even. By Lemma 1, we conclude that q is even. So, both p and q are even, a contradiction with the earlier assumption that p and q are coprime. □